HEAT CONDUCTION IN A UNIFORM LAYER WEAKENED BY A RECTANGULAR CUT-OUT

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Formulas are obtained by solving a two dimensional boundary-value problem of stationary heat conduction, which enable one to find an estimate of the weakening of heat insulation of of layer by a rectangular cut-out.

The weakening of the heat shielding properties of a layer due to the presence of various cut-outs can be described by the ratio $m = q/q_0$ of conduction heat fluxes, in the presence or absence of a cut-out. The value of m is found in the case of rectangular cut-out.

For example, let there be a rectangular cut-out in an infinite strip of uniform layer adjoining one of its boundaries (Fig. 1a), and let the temperature on the layer boundaries be constant ($t = 0$ on the boundary A'M' and $t = t_0$ on the boundary AM). For these stationary conditions one has a boundary-value problem for the Laplace's equation with boundary conditions of the first kind [1]:

$$\Delta w(z) = 0; \quad \operatorname{Re} w = \begin{cases} 0, & z \in A'M', \\ t_0, & z \in AM. \end{cases} \tag{1}$$

In the above $w(z) = t + i\psi$ is the complex heat potential; $z = x + iy$; $\psi(z)$ is the heat-stream function.

The above problem can be reduced by conformal mapping to the Dirichlet problem in the upper half-plane of the variable $\zeta = \xi + i\eta$. If the layer boundary is mapped into the real ξ axis and the points $z = 0$; $+\infty$; ic (Fig. 1a) are mapped into the points $\xi = 0$; 1; $+\infty$ (Fig. 1b) then by the Schwarz formula one obtains

$$w = \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{t(\xi)\, d\xi}{\xi - \zeta} = \frac{t_0}{\pi i} \ln \frac{\zeta - 1}{\zeta + 1}. \tag{2}$$

The heat flux which passes through the portion RT of the upper boundary of the layer in the unit of time is $q = -\lambda \int_{RT} \frac{\partial t}{\partial y}\, dx = -\lambda \int_{RT} d\psi = -2\lambda\psi(\xi_0)$. If there is no cut-out then the heat flow through that portion is $q = q_0 = -2\lambda t_0 a/h$. Therefore

$$m = \frac{1}{\alpha} \operatorname{arcth} \xi_0 = \frac{1}{\alpha} \operatorname{arsh} \frac{1}{\sqrt{\xi_0^2 - 1}}, \tag{3}$$

where $\xi_0 > 1$ is the image of the end point T; $\alpha = \pi a/2h$.

To find ξ_0 one uses the Schwarz–Christoffel formula [2] for the mapping function:

$$\frac{\pi z}{2h} = C \int_0^{\zeta} \sqrt{\frac{\zeta^2 - s^2}{\zeta^2 - p^2}} \frac{d\zeta}{1 - \zeta^2}, \tag{4}$$

where s and p are parameters and $C = \sqrt{(1 - p^2)/(1 - s^2)}$ is the mapping constant. Bearing in mind the change of sign of the square root in (4) on different portions of the real ξ axis one has

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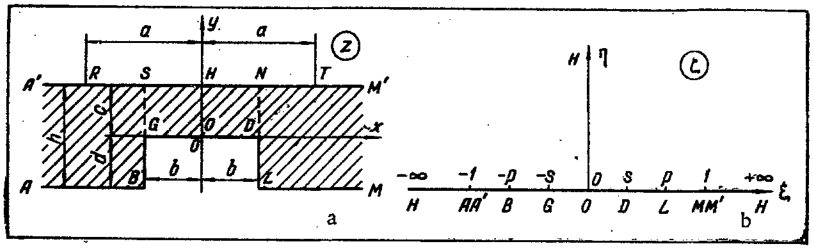

Fig. 1. The region of the uniform layer (a) weakened by a rectangular cut-out and the corresponding region in the plane of the parametric variable (b).

$$\beta = \frac{\pi b}{2h} = C \int_0^s \sqrt{\frac{s^2 - \xi^2}{p^2 - \xi^2}} \frac{d\xi}{1 - \xi^2}; \quad \gamma = \frac{\pi d}{2h} = C \int_s^p \sqrt{\frac{\xi^2 - s^2}{p^2 - \xi^2}} \frac{d\xi}{1 - \xi^2}. \tag{5}$$

The elliptic integrals in (5) can be brought to their standard form by respective substitutions: $\zeta = sl$; $\zeta = \sqrt{p^2 - l^2(p^2 - s^2)}$. Then

$$\beta = \frac{C}{p}\left[K - (1 - s^2)\Pi\left(\frac{\pi}{2}; -s^2; k\right)\right] = EF(\varphi; k) - KE(\varphi; k) + CksK, \tag{6}$$

$$\gamma = \frac{C}{p}\left[\frac{1}{C^2}\Pi\left(\frac{\pi}{2}; \frac{p^2 - s^2}{1 - p^2}; k'\right) - K'\right] = (E' - K')F(\varphi; k) + K'E(\varphi; k) - CksK. \tag{7}$$

Here and below K; K'; $F(\varphi; k)$; E; E'; $E(\varphi; k)$; $\Pi(\varphi; n; k)$ is the generally adopted notation for complete and incomplete elliptic integrals of the first, second, and third kinds [4] (the prime indicates that the integral is taken over the complementary module $k' = \sqrt{1 - k^2}$); $k = s/p$; $\varphi = \arcsin p$.

The relations (6) and (7) are now multiplied by K' and K, respectively, and the results are added. Hence by using the Legendre relation [3] one obtains

$$p = \operatorname{sn}(u; k); \; u = F(\varphi, k) = \frac{b}{h}K' + \frac{d}{h}K,$$
$$E(\varphi, k) = Ck^2p + [Ed - b(E' - K')]/h. \tag{8}$$

To find the value $\xi_0 = p/\sin\theta$ one expresses the length of the portion under consideration in terms of the parameters p and s by using (4):

$$\alpha = C\int_\infty^{\xi_0} \sqrt{\frac{\xi^2 - s^2}{\xi^2 - p^2}} \frac{d\xi}{\xi^2 - 1} = \frac{C}{p}[F(\theta; k) - (1 - s^2)\Pi(\theta; -s^2; k)] + g(\xi_0), \tag{9}$$
$$g(\xi_0) = \operatorname{arth}\sqrt{\frac{\xi_0^2(1 - p^2)(1 - s^2)}{(\xi_0^2 - p^2)(\xi_0^2 - s^2)}}.$$

For $p \approx 1$ one has $(K'/K \approx c/b)$ $\alpha \approx \beta + g(\xi_0)$, $\xi_0^2 = 1/2(\mu + \sqrt{\mu^2 - 2p^2s^2})$, $\mu = p^2 + s^2 + (1 - p^2)(1 - s^2)\operatorname{cth}^2(\alpha - \beta)$.

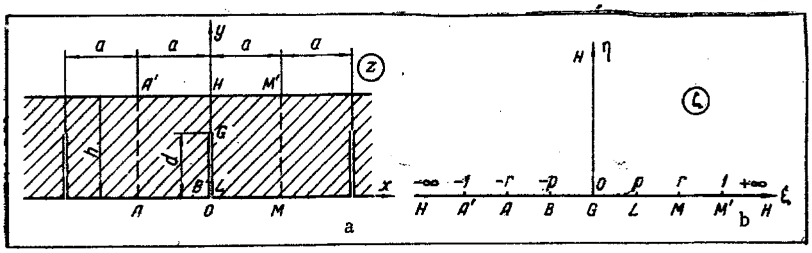

Fig. 2. A portion of a uniform layer (a) weakened by a periodic sequence of slit-like cut-outs and the region in the plane of the parametric variable corresponding to (b).

TABLE 1. Weakening of the Heat Shielding Properties of a Layer Depending on Geometric Parameters a/h and d/h of a Rectangular Cut-Out

a/h	1/3			0,5			1		
d/h	1/3	0,5	0,8	1/3	0,5	0,8	1/3	0,5	0,8
m_0 by formula (10)	1,140	1,362	2,558	1,125	1,317	2,236	1,085	1,207	1,722
m_0 by formula (20)	1,239	1,547	2,660	1,180	1,415	2,429	1,092	1,219	1,746

In the case of $b \to 0$ ($s \to 0$; $k \to 0$) when the cut-out approaches a slit the formulas (8)-(9) simplify to

$$p = \sin\gamma;\ \alpha = \operatorname{arth}\sqrt{\frac{1-p^2}{\xi_0^2 - p^2}};\quad \xi_0 = \sqrt{p^2 + (1-p^2)\operatorname{cth}^2\alpha}.$$

In accordance with (3) one obtains for the case $b = 0$,

$$m = m_0 = \frac{1}{\alpha}\operatorname{arsh}\left(\frac{\operatorname{sh}\alpha}{\cos\gamma}\right). \tag{10}$$

If, in accordance with the formula (10), the inhomogeneity of the thermal field on the boundaries of the cut-out is taken into account and the heat flux directly above the cut-out is considered as uniform ($q_{SN} = -2\lambda t_0 b/c$) then for the entire RT portion of the upper boundary of the layer one has approximately

$$m = \frac{bh}{ac} + \left(1 - \frac{b}{a}\right) m_0. \tag{11}$$

Carrying out the calculations with the aid of tables in [4] and using the formulas (4)-(9) for $d/h = 0.835$; $a/h = 0.36$; $b/h = 0.036$ (to which there corresponds $k^2 = 0.31$; $\varphi = 80°$; $\theta = 70°$) results in $\xi_0 = 1.05$ and $m = 3.292$ whereas by using the approximate formula (11) one finds that the estimate is equal to $m = 3.092$.

If the insulating layer has several cut-outs, the weakening can be estimated in this case by using the formula (11). However, the effect of the neighboring cut-outs should be taken into account.

Let us assume that there is a periodic sequence of slit-like cut-outs positioned on the side of the lower boundary of the layer, the step being $2a$. It suffices to consider the thermal field on a portion with a single cut-out (Fig. 2a) since the boundary lines AA' and MM' for this portion are the lines of the heat flux. The region A'ABGLMM' (Fig. 2a) is mapped on the upper half-plane of the variable ζ (Fig. 2b).

By using the Schwarz–Christoffel formula one obtains

$$z - (a + ih) = C_1 \int_1^{\zeta} \frac{\zeta d\zeta}{\sqrt{(\zeta^2 - r^2)(\zeta^2 - p^2)(\zeta^2 - 1)}} = \frac{C_1 F(\varphi; k)}{\sqrt{1-p^2}}, \tag{12}$$

where $k^2 = (r^2 - p^2)/(1 - p^2)$; $\varphi = \arcsin \varkappa$; $\varkappa^2 = (\zeta^2 - 1)/(\zeta^2 - r^2)$; $C_1 = \text{const}$.

The point $z = ih$ is mapped into $\zeta = +\infty$ ($\varkappa = 1$), that is, $a = -C_1 K/\sqrt{1-p^2}$. Therefore,

$$\left(1 + \frac{ih - z}{a}\right) K = F(\varphi, k). \tag{13}$$

The formula (13) is now inverted and subsequent transformations are expressed in terms of the Jacobi elliptic functions; this results in

$$\varkappa = \operatorname{sn}\left[\left(1 + \frac{ih - z}{a}\right) K;\ k\right]. \tag{14}$$

The condition (14) is satisfied at the point $z = a$ ($\zeta = r$; $\varkappa = +\infty$) if $h/a = K'/K$. Then

$$\varkappa = \operatorname{sn}\left[K + iK' - \frac{z}{a} K;\ k\right] = \frac{1}{k}\operatorname{dc}\left[\frac{z}{a} K;\ k\right]. \tag{15}$$

Hence by taking into account the correspondence between the points $z = id$ and $\zeta = 0$ ($\varkappa = 1/r$) and by using the Jacobi imaginary mapping [3] one obtains

$$r = \mathrm{dn}\left[\left(1 - \frac{d}{h}\right)K';\ k'\right]. \tag{16}$$

If one employs (16) and transforms (15) the mapping function can be represented in the form

$$\zeta = \sqrt{\frac{1 - r^2\varkappa^2}{1 - \varkappa^2}} = r\sqrt{1 - \mathrm{cn}^2\left(\frac{d}{h}K';\ k'\right)\mathrm{cn}^2\left(\frac{z}{a}K;\ k'\right)}. \tag{17}$$

Hence, since $z = 0$ and $\zeta = p$ are the corresponding points one has

$$p = r\,\mathrm{sn}\left(\frac{d}{h}K';\ k'\right).$$

Since the mapping parameters p and r are now known, the solution can be calculated with the aid of the Keldysh–Sedov formulas [2] for a mixed problem in the upper half-plane. It is, however, more convenient to consider the complex-potential plane w in which a rectangle with the sides ω_1 and $i\omega_2$ corresponds to the portions under consideration. The lateral sides of this rectangle are lines of the thermal stream; the upper and lower sides corresponding to the layer boundaries for that portion are isotherms ($t = 0$, $t = t_0$): $q = -2\lambda t_0(\omega_1/\omega_2)$. The rectangle with the sides ω_1 and $i\omega_2$ canbe regarded as a uniform portion of the layer which as regards thermal resistance is equivalent to the original portion with a cut-out in the z plane. The mapping of this rectangle into the upper halfplane (Fig. 2b) is known [2]:

$$\zeta = C_2\,\mathrm{sn}\left[\frac{K(k_0)}{\omega_1}w;\ k_0\right] \qquad (C_2 = \mathrm{const}). \tag{18}$$

In the above $C_2 = k_0 = r$;

$$\frac{\omega_1}{\omega_2} = \frac{K(r)}{K'(r)}. \tag{19}$$

Thus the weakening of the layer by slit-like cut-out with the neighboring cut-outs taken into account is given by

$$m_0 = \frac{hK(r)}{aK'(r)}, \tag{20}$$

where r is given by the formula (16); for $d/h = 0.5$ one has $r = \sqrt{k}$.

In Table 1 the results are given of calculations of m_0 carried out with the aid of tables given in [5] by the formula (10) and (20), that is, with or without the neighboring cut-outs taken into account.

Thus, for $a/h > 0.5$ and $d/h < 0.8$ one can use the formula (10) with an error which is less than 10% when m_0 is calculated for a layer weakened by several rectangular cut-outs.

NOTATION

t, temperature; q, heat flow; b, c, d, h, geometric dimensions of cut-out; $2a$, width of layer portion under consideration; λ, coefficient of heat conduction.

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